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THE EFFICIENCY OF DECENTRALIZED AND CENTRALIZED MARKETS FOR LEMONS

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Abstract

In markets with adverse selection, when average quality is low and frictions are small decentralized trade produces a greater surplus than predicted by the competitive model: under decentralized trade some high-quality units of the good trade whereas, due to the "lemons problem," only low-quality units trade in the competitive equilibrium. This suggests a reason why these markets are often decentralized. Remarkably, under some conditions payoffs are competitive as frictions vanish, even though all qualities trade.

Keywords: Quality Uncertainty, Adverse Selection, Markets for Lemons, Decentralized Markets, Bargaining and Matching, Efficency.

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1 Introduction

Markets differ in the degree in which trade is centralized. Call markets,[1] for example, are highly centralized and all trade takes place at a single price (the market clearing price). In contrast, in housing, labor, or used car markets, trade is highly decentralized, and prices are determined by bilateral bargaining between buyers and sellers, and may differ between trades. The competitive model abstracts away from these institutional aspects, thus providing a model suitable, in principle, for the study of both centralized and decentralized markets. Nonetheless, the assumption that prices are market clearing seems more appropriate for highly centralized markets than for decentralized ones, which raises the question of whether decentralized markets yield competitive outcomes. Indeed, it has been shown that in markets for homogenous goods decentralized trade tends to yield competitive outcomes when trading frictions are small, whether bargaining is under complete information (see, e.g., Gale (1987) or Binmore and Herrero (1988)) or under incomplete information (see, e.g., Serrano and Yosha (1996) or Moreno and Wooders (1999)).[2]

We study a simple market with adverse selection, and show that when trading frictions are small decentralized trade may produce superior outcomes (i.e., a greater surplus) than predicted by the competitive model: in a decentralized market high-quality units of the good trade when average quality is low while, due to the "lemons problem," only low-quality units trade in the competitive equilibrium. Thus, when frictions are small the additional gains realized from trading these high-quality units more than off-set the cost of the delay incurred for low-quality units under decentralized trade. The superiority of decentralized trade when average quality is low suggests why trade is often decentralized in markets with adverse selection.[3]

The market for lemons we study is a version of Akerlof's (1970) where the traded

[1] Call markets are used to set opening prices on the NYSE among others.

[2] There are some exceptions to these conclusions; see Rubinstein and Wolinsky (1985).

[3] Of course, another reason why trade is decentralized in markets with adverse selection is that the goods often differ in their observable characteristics. Heterogeneity in observable characteristics (e.g., no two homes are identical) reduces the competitiveness of markets. In order to focus on the effects of (heterogenous) unobservable quality, we assume that goods are identical in terms of their observable characteristics.

good is of either high or low quality. The proportion of high quality units in the market (q^H) determines the properties of the competitive equilibria: when the expected value to a buyer of a randomly selected unit of the good ($u(q^H)$) is above the cost of high quality (c^H) there are multiple competitive equilibria (see Figure 1a): there are equilibria in which all units of both qualities trade, but there is also an equilibrium in which all low quality and some high quality units trade, and there is an equilibrium in which only low quality units trade. When it is below (i.e., when $u(q^H) < c^H$) then there is a unique competitive equilibrium in which only low quality units trade (see Figure 1b).

Figure 1 goes here.

In our model of decentralized trade, at each period every agent in the market has a positive probability of meeting an agent of the opposite type. Once matched, the buyer makes a take-it-or-leave-it price offer to the seller.[4] If the seller accepts, then they trade at the offered price and both agents exit the market. If the seller rejects the offer, then both the buyer and the seller remain in the market at the next period. Traders are impatient, and discount future gains to trade. The discounting of future gains and the time-consuming nature of matching constitute trading "frictions." We consider both the stationary entry case, where the supply and demand curves in Figure 1 represent the (stationary) flows of agents entering the market at each date, and the one-time entry case, where the supply and demand curves in Figure 1 represent the agents entering at the market open.

For the stationary entry case, we study the (stationary) equilibrium when frictions are small. We show that the welfare properties of equilibrium are determined by the proportion of the entering sellers with high quality units: when $u(q^H) \geq c^H$, then the surplus realized under decentralized trade is less than the surplus in the most

[4] Modeling a decentralized market requires specifying the trading rules. We chose this particular trading rule because it simplifies the analysis and allows us to focus on evaluating the differences between centralized and decentralized trade. Wilson (1980) studies the impact on market outcomes of different price setting institutions, and Bester (1993) studies which pricing institutions may emerge in markets with adverse selection.

efficient competitive equilibrium (i.e., the competitive equilibrium at which all units of both qualities trade), but greater than the surplus in the least efficient competitive equilibrium (i.e., the competitive equilibrium at which only low-quality units trade). When $u(q^H) < c^H$, then the surplus realized under decentralized trade is greater than the surplus in the (unique) competitive equilibrium. Remarkably, decentralized trade yields competitive payoffs as frictions vanish.

For the one-time entry case, the analysis of the infinite horizon version of the model becomes very complex. As an example, we study a market that operates over two periods. We show that the properties of the market equilibrium depend upon whether the initial proportion of high quality units, q^H, is above or below a critical threshold, q^*. When $q^H > q^*$ then all matched buyers and sellers trade at a price of c^H. When $q^H < q^*$, then both high and low quality units trade with positive probability, and there is trade at more than one price. As for the surplus, our results are illustrated in Figure 2 which shows (*i*) the surplus at the competitive equilibria and (*ii*) the surplus realized under decentralized trade as frictions vanish. The figure reveals that if $q^H > q^*$ then the surplus realized under decentralized trade is (asymptotically) the same as the surplus in the most efficient competitive equilibrium. If $u(q^H) < c^H$, then the surplus realized under decentralized trade is (asymptotically) greater than that realized in the competitive equilibrium. For intermediate values of q^H, where $u(q^H) > c^H$ but $q^H < q^*$, the surplus realized under decentralized trade is smaller than the surplus in the most efficient competitive equilibrium, but greater than the surplus in the least efficient competitive equilibrium.

Figure 2 goes here.

Taken together, our results for stationary and one-time entry show that when average quality is low and frictions are small, then the gains realized are higher under decentralized trade than in the competitive equilibrium. Further, when frictions are small the gains to trade are higher under decentralized trade than in the least efficient competitive equilibrium, whether average quality is high or low.

Following Akerlof's (1970) seminal paper, the literature studying markets with adverse selection has become too large for us to attempt to survey here (see, e.g., Wilson (1980), Bond (1982), Kim (1985), Bester (1993), Gale (1996), etc.). In a concurrent paper, Blouin (2001) studies a decentralized market for lemons in a model which differs from ours in that (i) the probability of matching is set to one, (ii) it assumes that average quality is low (i.e., that $u(q^H) < c^H$), and more significantly (iii) it allows only one of three exogenously given prices to emerge from bargaining.[5] Perhaps not surprisingly, Blouin obtains results quite different from ours: he finds, for example, that each type of trader obtains a positive payoff (and therefore payoffs are not competitive) even as frictions vanish. This result, which is at odds with our finding when entry is stationary that payoffs are competitive as frictions vanish, seems to be driven by the exogeneity of prices in Blouin's model. (In our model, prices are determined endogenously without prior constraints.) In addition, the ranking of surplus under centralized and decentralized trade depends on how the three prices in his model are chosen; since these prices do not seemly relate to economic primitives, these results are inconclusive. For the one-time entry case Blouin (2001) obtains results in a model with an infinite horizon. We do not provide a comparable analysis.

Our paper is organized as follows. In Section 2 we describe the lemons market we study and its competitive equilibria. In Section 3 we introduce our model of decentralized trade and establish results for one-time and stationary entry. We conclude in Section 4 with a discussion. The proofs are presented in the Appendix.

2 A Market for Lemons

Consider a market for an indivisible commodity which can be of either high or low quality. There is a continuum of buyers and sellers present in equal measures (that we normalize to one). A proportion $q^H \in (0, 1)$ of sellers are endowed with a (single) unit of high-quality good, whereas the remaining proportion $q^L = 1 - q^H$ of sellers are endowed with a unit of low-quality good. A seller knows the quality of his good, but prior to purchase quality is unobservable by buyers. We refer to the sellers endowed with a unit of high (low) quality good as high (low) quality sellers. Preferences are characterized by values and costs: the cost to a seller of a unit of the good when it

[5] This three-price set up was introduced by Wolinsky (1990).

is of high (low) quality is c^H (c^L); the value to a buyer of a unit of the good when it is of high (low) quality is u^H (u^L). Each type of good is valued more highly by buyers than by sellers (i.e., $u^H > c^H$ and $u^L > c^L$), and both buyers and sellers value high quality more than low quality (i.e., $u^H > u^L$ and $c^H > c^L$). Also we assume that $c^H > u^L$, since otherwise the lemon's problem doesn't arise. Thus, we assume throughout that $u^H > c^H > u^L > c^L$.

When a buyer and a seller trade at the price p, the buyer obtains a utility of $u-p$ and the seller obtains a utility of $p-c$, where $u = u^H$ and $c = c^H$ if the unit traded is of high quality, and $u = u^L$ and $c = c^L$ if it is of low quality. A buyer or a seller who does not trade obtains a utility of zero. For $q \in [0,1]$ we write $u(q) = qu^H + (1-q)u^L$ for the expected value to a buyer of a randomly selected unit of the good when a proportion q of all units are of high quality.

Properties of Competitive Equilibria

We begin by characterizing competitive outcomes, which we take to be the benchmark for centralized trade. In the competitive model the behavior of buyers and sellers is described by aggregate supply and demand correspondences. Denote by p the market price. If $p > c^\tau$ then all τ-quality sellers supply a unit. Conversely, if $p < c^\tau$ then no τ-quality seller supplies. Hence, for $\tau \in \{H, L\}$, aggregate supply of the τ-quality good $z^\tau : \mathbb{R}_+ \twoheadrightarrow [0,1]$ is given by

$$z^\tau(p) = \begin{cases} \{q^\tau\} & \text{if } p > c^\tau \\ [0, q^\tau] & \text{if } p = c^\tau \\ \{0\} & \text{if } p < c^\tau. \end{cases}$$

The aggregate demand, $z^B : \mathbb{R}_+ \times [0,1] \twoheadrightarrow [0,1]$, depends upon both the price $p \in \mathbb{R}_+$ and the fraction of the units supplied that are of high quality[6] $\mu \in [0,1]$, and

[6]Since high and low quality sellers supply at different prices, the proportion of the units supplied which are of high quality, μ, may differ from the proportion of sellers who own a unit of high quality good, q^H.

is given by

$$z^B(p,\mu) = \begin{cases} \{0\} & \text{if } p > u(\mu) \\ [0,1] & \text{if } p = u(\mu) \\ \{1\} & \text{if } p < u(\mu). \end{cases}$$

A *competitive equilibrium (CE)* is a collection $e = (p, \mu, z^B, z^H, z^L)$ such that

$(CE.1)$ $z^B \in z^B(p,\mu)$, $z^H \in z^H(p)$ and $z^L \in z^L(p)$,

$(CE.2)$ $\mu = z^H/(z^H + z^L)$ whenever $z^H + z^L > 0$, and

$(CE.3)$ $z^B - z^L - z^H = 0$.

In a CE traders behave optimally $(CE.1)$, their expectations are rational $(CE.2)$, and the market clears $(CE.3)$.

A competitive equilibrium provides an aggregate description of the final allocation; that is, it specifies a transaction price and volumes (i.e., measures) of trade of high and low quality good. The surplus realized in a CE is given by

$$S^C = z^H(u^H - c^H) + z^L(u^L - c^L).$$

Given values and costs, we denote by $S^C(q^H)$ the set of surpluses that are realized in the competitive equilibria of the market where a proportion q^H of the sellers are high quality – as we shall see, equilibrium may not be unique.

The properties of competitive equilibria are determined by the relation between the expected value to a buyer of a randomly selected unit of the good, $u(q^H)$, and the cost for high-quality sellers, c^H. If $u(q^H) \geq c^H$, then there are multiple competitive equilibria (see Figure 1.a), whereas if $u(q^H) < c^H$, then there is a unique competitive equilibrium (see Figure 1.b). Proposition 1 summarizes the properties of these equilibria. This properties are well known, and are given without proof.

Proposition 1.

$(P1.1)$ *If* $u(q^H) \geq c^H$*, then there are CE in which all sellers trade (at a price* $p \in [c^H, u(q^H)]$*), as well as a CE in which all low-quality sellers and some (but not all) high-quality sellers trade (at a price* $p = c^H$*), and a CE in which only low-quality sellers trade (at the price* $p = u^L$*). Thus,*

$$q^L(u^L - c^L) = \inf S^C(q^H) < \sup S^C(q^H) = q^H(u^H - c^H) + q^L(u^L - c^L).$$

(P1.2) If $u(q^H) < c^H$, then there is a unique CE. In this equilibrium only low-quality sellers trade (at the price $p = u^L$). Thus,

$$\inf S^C(q^H) = \sup S^C(q^H) = q^L(u^L - c^L).$$

A graph of the mapping $S^C(q^H)$ is given in Figure 2.

3 A Decentralized Market for Lemons

Consider now a market for lemons in which trade is decentralized. The market operates for a set of consecutive periods T, where T may be finite or infinite. (If T is finite, we abuse notation slightly and write $T = \{0, 1, \ldots, T\}$.) Each period $t \in T$ a measure q_t^τ of τ-quality sellers and a measure $q_t^B = q_t^H + q_t^L$ of buyers enter the market; then, every buyer (seller) in the market meets a randomly selected seller (buyer) with probability α, where $\alpha \in (0, 1)$. A matched buyer proposes a price at which to trade. If the proposed price is accepted by the seller, then the agents trade at that price and both leave the market. If the proposed price is rejected by the seller, then the agents remain in the market at the next period. An agent who is unmatched in the current period also remains in the market at the next period. Agents discount utility at a common rate $\delta \in (0, 1]$. An agent observes only the outcomes of his own matches.

A *strategy for a buyer* is a sequence of price offers $p = \{p_t\}_{t \in T}$, where $p_t \in \mathbb{R}_+$ is the price offer made if matched at date t. A *strategy for a seller* is a sequence $r = \{r_t\}_{t \in T}$ of reservation prices, where $r_t \in \mathbb{R}_+$ is the smallest price he accepts at date t.[7] A *strategy distribution* is a collection $s = [(p^{B_i}; \lambda^{B_i})_{i=1}^{n^B}, (r^{H_i}; \lambda^{H_i})_{i=1}^{n^H}, (r^{L_i}; \lambda^{L_i})_{i=1}^{n^L}]$, where $\lambda^{B_i} > 0$ is the proportion of buyers using the strategy p^{B_i}, $\lambda^{\tau_i} > 0$ is the proportion of type $\tau \in \{H, L\}$ sellers using strategy r^{τ_i}, and n^τ is the countable number of

[7] Price offers are "unconditional" since a buyer doesn't know whether the seller he is matched with is high or a low quality. Also, we consider only strategies in which a trader does not condition his actions in the current match on the history of his prior matches, but this restriction is inconsequential. Since a trader only observes the outcomes of his own matches, his decision problem is the same regardless of his history in prior matches – see Osborne and Rubinstein (1990), pp. 154-162.

distinct strategies used by a positive measure of type $\tau \in \{B, H, L\}$ traders. Thus, $\sum_{i=1}^{n^B} \lambda^{B_i} = \sum_{i=1}^{n^H} \lambda^{H_i} = \sum_{i=1}^{n^L} \lambda^{L_i} = 1$.

Laws of Motion

Let $[(p^{B_i}; \lambda^{B_i})_{i=1}^{n^B}, (r^{H_i}; \lambda^{H_i})_{i=1}^{n^H}, (r^{L_i}; \lambda^{L_i})_{i=1}^{n^L}]$ be a strategy distribution. For $\tau \in \{B, H, L\}$ and $k \le n^\tau$ let $\lambda_t^{\tau_k}$ denote the proportion of traders following the k-th type τ strategy out of the total measure of traders of type τ in the market at time t. This proportion can be computed for $t \in T$ as

$$\lambda_{t+1}^{\tau_k} = \frac{\lambda_t^{\tau_k}(1 - \alpha Z_t^{\tau_k})}{\sum_{l=1}^{n^\tau} \lambda_t^{\tau_l}(1 - \alpha Z_t^{\tau_l})}, \tag{1}$$

where $Z_t^{\tau_k}$ denotes the probability of trade for a type τ agent following the k-th type τ strategy who is matched at t. (If there is an initial period, $0 \in T$, then we take $\lambda_0^{\tau_k}$ to be given by the initial strategy distribution, i.e., $\lambda_0^{\tau_k} = \lambda^{\tau_k}$.) The probability $Z_t^{\tau_k}$ can be computed as follows: Denote by $I : \mathbb{R}_+^2 \to \{0, 1\}$ the *indicator function* whose value is $I(x, y) = 1$ if $x \ge y$, and $I(x, y) = 0$ otherwise. For sellers of type $\tau \in \{H, L\}$, this probability is given by

$$Z_t^{\tau_j} = \sum_{i=1}^{n^B} \lambda_t^{B_i} I(p_t^{B_i}, r_t^{\tau_j}). \tag{2}$$

For buyers it is given by

$$Z_t^{B_i} = \sum_{\tau \in \{H,L\}} \mu_t^\tau \sum_{j=1}^{n^\tau} \lambda_t^{\tau_j} I(p_t^{B_i}, r_t^{\tau_j}), \tag{3}$$

where for $\tau \in \{H, L\}$, μ_t^τ is the proportion of type τ sellers out the "stock" of sellers in the market at time t; i.e., μ_t^τ is the ratio

$$\mu_t^\tau = \frac{K_t^\tau}{K_t^H + K_t^L}, \tag{4}$$

where K_t^τ, the stock of sellers of type $\tau \in \{H, L\}$ in the market at time t, is given by

$$K_t^\tau = (1 - \alpha \sum_{j=1}^{n^\tau} \lambda_{t-1}^{\tau_j} Z_{t-1}^{\tau_j}) K_{t-1}^\tau + q_t^\tau. \tag{5}$$

Since the measure of buyers and sellers entering – and leaving – the market each period are identical, the stock of buyers at $t \in T$ is $K_t^B = K_t^H + K_t^L$. Again, if there is an initial period $0 \in T$, then we take $K_0^\tau = q_0^\tau$ for $\tau \in \{B, H, L\}$.

Value Functions

Given a strategy distribution $[(p^{B_i};\lambda^{B_i})_{i=1}^{n^B},(r^{H_i};\lambda^{H_i})_{i=1}^{n^H},(r^{L_i};\lambda^{L_i})_{i=1}^{n^L}]$, the expected utility at time $t\in T$ of a buyer following the strategy p^{B_i} is

$$V_t^{B_i}=\alpha\sum_{\tau\in\{H,L\}}\mu_t^\tau\sum_{j=1}^{n^\tau}\lambda_t^{\tau_j}[I(p_t^{B_i},r_t^{\tau_j})(u^\tau-p_t^{B_i})+(1-I(p_t^{B_i},r_t^{\tau_j}))\delta V_{t+1}^{B_i}]+(1-\alpha)\delta V_{t+1}^{B_i}. \tag{6}$$

The expected utility of a seller of type $\tau\in\{H,L\}$ following the strategy r^{τ_j} is

$$V_t^{\tau_j}=\alpha\sum_{i=1}^{n^B}\lambda_t^{B_i}[I(p_t^{B_i},r_t^{\tau_j})(p_t^{B_i}-c^\tau)+(1-I(p_t^{B_i},r_t^{\tau_j}))\delta V_{t+1}^{\tau_j}]+(1-\alpha)\delta V_{t+1}^{\tau_j}. \tag{7}$$

When T is finite, these expected utilities are computed using $V_{T+1}^{\tau_k}=0$ for each $\tau\in\{B,H,L\}$ and $k\le n^\tau$.

Equilibrium

A strategy distribution $[(p^{B_i};\lambda^{B_i})_{i=1}^{n^B},(r^{H_i};\lambda^{H_i})_{i=1}^{n^H},(r^{L_i};\lambda^{L_i})_{i=1}^{n^L}]$ is a *market equilibrium* if for each $t\in T$, $i\in\{1,\dots,n^B\}$, $j\in\{1,\dots,n^\tau\}$ and $\tau\in\{H,L\}$:

$(ME.\tau)$ $r_t^{\tau_j}-c^\tau=\delta V_{t+1}^{\tau_j}$, and

$(ME.B)$ $p_t^{B_i}\in\arg\max_{p\in\mathbb{R}_+}\sum_{\tau\in\{H,L\}}\mu_t^\tau\sum_{j=1}^{n^\tau}\lambda_t^{\tau_j}[I(p,r_t^{\tau_j})(u^\tau-p)+(1-I(p,r_t^{\tau_j}))\delta V_{t+1}^{B_i}]$.

Condition $ME.\tau$ ensures that the reservation price of each type τ seller makes him indifferent between accepting or rejecting an offer of his reservation price. Condition $ME.B$ ensures that buyer price offers are optimal.

A straightforward implication of the definition of market equilibrium is that traders of the same type have identical expected utilities and that, as a direct consequence, the reservation prices and probabilities of trade are the same for sellers of the same type. Formally, we have:

Remark 1. *For $t\in T$:*

$(R1.B)$ $V_t^{B_i}=V_t^B$ for each $i\le n^B$, and

$(R1.\tau)$ $V_t^{\tau_j}=V_t^\tau$, $r_t^{\tau_j}=r_t^\tau$ and $Z_t^{\tau_j}=Z_t^\tau$ for each $\tau\in\{H,L\}$ and each $j\le n^\tau$.

We therefore restrict attention to strategy distributions in which all sellers of the same type follow the same strategy (i.e., where $n^H=n^L=1$). As we shall see, however, allowing buyers to follow different strategies is necessary to guarantee existence of a market equilibrium.

3.1 One-Time Entry

In this section we analyze a decentralized market for lemons with one-time entry; i.e., such that $q_0^\tau = q^\tau > 0 = q_t^\tau$ for $\tau \in \{H, L, B\}$ and $t > 0$. Without loss of generality, we normalize the measure sellers and buyers to be one; i.e., $q^H + q^L = q^B = 1$. Also we consider a market that operates for two periods; i.e., such that $T = \{0, 1\}$. (The analysis of market that operates for a longer horizon is complex, as the proportion of high quality sellers in the market is not stationary.)

Proposition 2 below establishes the basic properties of market equilibria. These properties are determined by the relation between the initial proportion of high quality sellers q^H, the threshold $q^* = (c^H - c^L)/(u^H - c^L)$, and the magnitude of the market frictions (i.e., the discount factor δ and the matching probability α). The threshold q^*, which is the solution to the equation $q^* u^H + (1 - q^*) u^L - c^H = (1 - q^*)(u^L - c^L)$, is the proportion of high quality sellers that makes a buyer indifferent between offering c^H (when this offer is accepted by both types of sellers) or offering c^L (when this offer is accepted only by low quality sellers) in a one-shot take-it-or-leave it bargaining game with a randomly selected seller.

Proposition 2. *Consider a decentralized market for lemons for which $T = \{0, 1\}$, and $q_0^\tau = q^\tau > 0 = q_1^\tau$ for $\tau \in \{H, L, B\}$. In every market equilibrium we have $r_t^H = c^H > r_t^L$ for $t \in \{0, 1\}$. In addition:*

(P2.1) If $q^H > q^*$, then in the unique market equilibrium all buyers offer r_t^H at each period. Thus, at each period all matched sellers trade.

(P2.2) If $q^H < q^*$ and δ and α are near one, then in every market equilibrium only price offers of r_0^L and of less than r_0^L are made by positive measures of buyers at $t = 0$, and only price offers of r_1^H and of r_1^L are made by positive measures of buyers at $t = 1$. Thus, at $t = 0$ only matched low-quality sellers trade with positive probability (but less than one); and at $t = 1$, matched low-quality sellers trade with probability one, and matched high-quality sellers trade with positive probability (but less than one). Moreover, reservation prices, price offers (except for rejected price offers), the probability of trade, and the expected utility of each trader are uniquely determined.

The implications of Proposition 2 are clear:

$(P2.1)$ If high-quality sellers are abundant, then there is a unique equilibrium in which buyers offer c^H (a competitive equilibrium price) in both periods, and every match ends with trade.

$(P2.2)$ If high-quality sellers are scarce and frictions are small, then buyers "mix" in both periods: In the first period a positive proportion (less than one) of buyers offer the low-quality-seller reservation price r_0^L (and those matched with a low-quality seller trade), and the remaining matched buyers offer prices below r_0^L (and do not trade). In the second period, a positive proportion (less than one) of buyers offer the reservation price of high-quality sellers $r_1^H = c^H$ (and trade), while the remaining matched buyers offer the reservation price of low-quality sellers $r_1^L = c^L$ (and only those matched with low-quality sellers trade). The exact equilibrium mixtures involved are provided in the proof of Proposition 2. Equilibrium is essentially unique (and asymmetric); specifically, the prices at which trade occurs and the measures of trade at each of these prices are uniquely determined.

Surplus

By Proposition 2, when frictions are small then traders' expected utilities are uniquely determined. Hence, the surplus realized in a market equilibrium is a function of the initial proportion of high-quality sellers in the market, and is given by

$$S^D(q^H) = V_0^B + q_0^H V_0^H + q_0^L V_0^L.$$

Proposition 3 compares the surplus under centralized trade (i.e., the competitive equilibrium surplus) and the surplus under decentralized trade, both when frictions are non-negligible and also as frictions vanish. To simplify our notation, we write $\tilde{S}^D(q^H)$ for $\lim_{\delta\to 1}\lim_{\alpha\to 1} S^D(q^H)$ and $\lim_{\alpha\to 1}\lim_{\delta\to 1} S^D(q^H)$, when both limits exist and coincide.

Proposition 3. *Consider a decentralized market for lemons for which $T = \{0, 1\}$, and $q_0^\tau = q^\tau > 0 = q_1^\tau$ for $\tau \in \{H, L, B\}$, and assume that δ and α are near one.*
$(P3.1)$ If $q^H > q^*$, then

$$\inf S^C(q^H) < S^D(q^H) < \tilde{S}^D(q^H) = \sup S^C(q^H).$$

$(P3.2)$ If $q^H < q^*$ and $u(q^H) > c^H$, then

$$\inf S^C(q^H) < S^D(q^H) < \tilde{S}^D(q^H) < \sup S^C(q^H).$$

$(P3.3)$ If $u(q^H) < c^H$, then

$$\inf S^C(q^H) = \sup S^C(q^H) < S^D(q^H) < \tilde{S}^D(q^H).$$

By Proposition 3 whether the surplus under decentralized trade is greater or less than the competitive equilibrium surplus depends upon the initial proportion of high quality sellers in the market: As Figure 2 illustrates, when $q^H > q^*$, the surplus realized under decentralized trade coincides (as frictions vanish) with the surplus in the most efficient CE, and is greater than the surplus in either of the other two CE. When $q^H < q^*$ and $u(q^H) > c^H$, the surplus realized under decentralized trade is less than the surplus realized in the most efficient CE, but it is greater than the surplus realized in the least efficient CE. When $u(q^H) < c^H$ and frictions are small, the surplus realized under decentralized trade is greater than the (unique) CE surplus. Thus, for every value of q^H, decentralized trade yields a greater surplus than the least efficient CE surplus (in which only low-quality units trade).

3.2 Stationary Entry

In this section we study equilibrium in a decentralized market for lemons that operates over an infinite number of periods, and where there is a constant flow of agents of each type entering the market each period, $q_t^\tau = q^\tau > 0$ for $\tau \in \{H, L, B\}$. Again without loss of generality we assume that $q^H + q^L = q^B = 1$.

We shall restrict attention to equilibria in stationary strategies, i.e., to strategies that are constant sequences. We can describe a stationary strategy for a buyer (seller of type $\tau \in \{H, L\}$) by a non-negative real number p^B (r^τ) indicating a price offer (reservation price) at each date $t \in T$. A (stationary) strategy distribution is therefore described by a collection $s = [(p^{B_i}, \lambda^{B_i})_{i=1}^{n^B}, r^H, r^L]$, where each λ^{B_i} is the proportion of buyers following the ith buyer strategy p^{B_i}, and n^B is the countable number of distinct strategies used by buyers.

We study the stationary equilibria of this market. A stationary equilibrium is a stationary strategy distribution that constitutes a market equilibrium (as defined above) and has the additional property that the measure of each type of trader in the market is constant over time. Formally, a stationary equilibrium is defined as follows:

Stationary Equilibrium

A stationary strategy distribution $[(p^{B_i}, \lambda^{B_i})_{i=1}^{n^B}, r^H, r^L]$ is a *stationary equilibrium* if there are stocks $(K^H, K^L) \in \mathbb{R}_+^2$, such that for each $i \in \{1, \dots, n^B\}$ and $\tau \in \{H, L\}$:

$(SE.\tau)$ $r^\tau - c^\tau = \delta V^\tau$, and

$(SE.B)$ $p^{B_i} \in \arg\max_{p \in \mathbb{R}_+} \sum_{\tau \in \{H,L\}} \mu^\tau [I(p, r^\tau)(u^\tau - p) + (1 - I(p, r^\tau))\delta V^{B_i}]$,

$(SE.K)$ $\alpha K^\tau \sum_{i=1}^{n^B} \lambda^{B_i} I(p^{B_i}, r^\tau) = q^\tau$.

Conditions $SE.\tau$ for $\tau \in \{H, L\}$ and $SE.B$ ensure that sellers and buyers behave optimally. Condition $SE.K$, which is just equation (5) with the time subscript removed, ensures that the measure of type τ sellers exiting the market each period equals the measure of type τ sellers entering each period. A consequence of $SE.K$ is that the proportion of type τ sellers (μ^τ), and the expected utility of type τ traders (V^τ), which are computed using equations (4), (6), and (7) with the time subscript removed, are stationary.

Properties of stationary Equilibrium

Given a stationary equilibrium, the *(flow) surplus* is the sum of the expected utilities of the flow of agents entering every period, i.e.,

$$S^F(q^H) = V^B + q^H V^H + q^L V^L.$$

When there is unique stationary equilibrium (see Proposition 4.3 below), it is meaningful to express flow surplus as a function of q^H. The properties of stationary equilibria are establish in Proposition 4.

Proposition 4. *Assume that $\delta < 1$ is near one.*

(P4.1) If $u(q^H) \geq c^H$, there is a stationary equilibrium in which all buyers offer c^H and all sellers accept offers of c^H. Moreover, in this stationary equilibrium as δ approaches one (i) the surplus approaches (from below) the surplus at the most efficient competitive equilibrium, and (ii) each trader's expected utility is the same as in the competitive equilibrium in which all units trade at the price c^H.

(P4.2) If $u(q^H) < c^H$, there is a stationary equilibrium in which a positive proportion of buyers offer c^H (which all sellers accept), a positive proportion offer u^L (which only low-quality sellers accept), and a positive proportion offer less than u^L (which all sellers reject). Moreover, in this stationary equilibrium as δ approaches one (i) the surplus approaches (from above) the surplus at the competitive equilibrium, and (ii) each trader's expected utility is the same the same as in the unique competitive equilibrium.

(P4.3) If $u^H - c^H > u^L - c^L$, the stationary equilibria described in (P4.1) and (P4.2) are unique (up to rejected price offers).

Proposition 4 establishes that the comparison of surplus under centralized and decentralized trade is determined by the relation between the expected value to a buyer of a randomly selected unit of the good, $u(q^H)$, and the cost for high-quality sellers, c^H. When $u(q^H) \geq c^H$ the surplus realized in a decentralized market with stationary entry is *smaller* than the surplus in the most efficient competitive equilibria (but greater than the surplus generated in the least efficient competitive equilibrium). When $u(q^H) < c^H$ and frictions are small (but not negligible), the surplus realized in a decentralized market with stationary entry is *greater* than the competitive surplus.

By Proposition 4, whether $u(q^H) \geq c^H$ or $u(q^H) < c^H$, as frictions vanish each trader obtains a competitive equilibrium payoff; when $u(q^H) \geq c^H$ then traders obtain the same payoff as in the competitive equilibrium in which all units trade at the price c^H. When $u(q^H) < c^H$, then traders obtain the same payoff as in the (unique) competitive equilibrium, with price u^L. This last result is remarkable since in the competitive equilibrium only low quality trades, while in the stationary equilibrium high-quality units also trade.

When the gains to trade for high-quality units is greater than the gains to trade for low-quality units (i.e., when $u^H - c^H > u^L - c^L$), the equilibria described in $(P4.1)$ and $(P4.2)$ are unique except that the price offers below u^L are not determined. This multiplicity of equilibria is irrelevant since prices offers of this kind are rejected.

An interesting implication of Proposition 4 is that in a stationary equilibrium the proportion of high-quality sellers in the market, μ^H, always satisfies $u(\mu^H) \geq c^H$: when the measure of high quality sellers entering the market, q^H, satisfies $u(q^H) \geq c^H$, then in the stationary equilibrium we have $\mu^H = q^H$; when $u(q^H) < c^H$, then the proportion of high-quality sellers in the market adjusts so that $u(\mu^H) = c^H$. Thus, in the later case, high quality sellers are present in the market in a higher proportion than they enter the market.[8]

4 Discussion

An interesting implication of our results is that, based on the surplus generated, neither centralized trade nor decentralized trade dominates the other: decentralized trade performs better (i.e., generate a larger surplus) than predicted by the competitive model when average quality is low and frictions are small, whereas centralized trade may produce superior outcomes when average quality is high. An obvious question to ask is how far is the surplus realized under these market structures from the surplus that can be realized by an *efficient mechanism* (i.e., a mechanism that maximizes the surplus over all incentive compatible and individually rational mechanisms). In our context, a mechanism is defined by a pair (p, Z), where $p = (p^H, p^L) \in \mathbb{R}^2_+$ and $Z = (Z^H, Z^L) \in [0, 1]^2$, specifying, for each quality report $\tau \in \{H, L\}$ of a seller, a money transfer p^τ (from the buyer to the seller), and a probability Z^τ that the seller transfers the good to the buyer.[9] An efficient mechanism is therefore a solution to the problem

$$\max_{(p,Z)\in\mathbb{R}^2_+\times[0,1]^2} q^H Z^H (u^H - c^H) + q^L Z^L (u^L - c^L)$$

subject to

$$p^\tau - Z^\tau c^\tau \geq p^\sigma - Z^\sigma c^\tau \text{ for each } \tau, \sigma \in \{H, L\}, \tag{$IC.\tau$}$$

$$q^H Z^H u^H + q^L Z^L u^L - (q^H p^H + q^L p^L) \geq 0, \tag{$IR.B$}$$

[8] Bond (1982) provides a test of the "lemons problem" by comparing the average quality of traded and non-traded goods. Our model predicts that the average quality in the stock of units that has not yet traded is greater than the average quality in the flow of units trading.

[9] By the Revelation Principle, we can restrict attention to "direct" mechanisms. Also note that there is no need for buyers to report to the mechanism since they have no private information.

$$p^{\tau} - Z^{\tau} c^{\tau} \geq 0 \text{ for each } \tau \in \{H, L\}. \tag{$IR.\tau$}$$

The constraint $IC.\tau$ guarantees that the mechanism is incentive compatible, i.e., it is optimal for a type τ seller to report his type truthfully. The constraints $IR.B$ and $IR.\tau$ guarantee that participating in the mechanism is individually rational for buyers and sellers; i.e., that no trader obtains a negative expected payoff.

It is straightforward to show that if $u(q^H) \geq c^H$, then every efficient mechanism satisfies $Z^H = Z^L = 1$, and generates a surplus, $S(q^H)$, given by

$$S(q^H) = q^H(u^H - c^H) + q^L(u^L - c^L).$$

When $u(q^H) < c^H$, however, then the efficient mechanism satisfies $Z^L = 1 > Z^H = q^L(u^L - c^L)/\left(c^H - c^L - q^H(u^H - c^L)\right)$, and therefore generates a surplus of

$$S(q^H) = q^H(u^H - c^H)\frac{q^L(u^L - c^L)}{c^H - c^L - q^H(u^H - c^L)} + q^L(u^L - c^L).$$

Figure 3 below provides graphs of the mappings $S(q^H)$, $\tilde{S}^D(q^H)$, and $S^C(q^H)$, illustrating the relation between the surpluses in a market with one-time entry.

Figure 3 goes here.

As Figure 3 shows, when $q^H > q^*$, we have $\tilde{S}^D(q^H) = \sup S^C(q^H) = S(q^H) > \inf S^C(q^H)$; that is, the efficient surplus is realized both under decentralized trade (as frictions vanish), and at the most efficient competitive equilibria (but not at the other competitive equilibria). When $u(q^H) < c^H$, however, we have $\inf S^C(q^H) = \sup S^C(q^H) < \tilde{S}^D(q^H) < S(q^H)$; that is, both centralized and decentralized trade generate a surplus below the efficient surplus (although decentralized trade performs better). For intermediate values, i.e., when $q^H < q^*$ but $u(q^H) > c^H$, we have $\sup S^C(q^H) = S(q^H) > \tilde{S}^D(q^H) > \inf S^C(q^H)$; that is, centralized trade may generate the efficient surplus, whereas decentralized trade generates less than the efficient surplus.[10]

[10] Gale (1996) studies the properties of the competitive equilibria of markets with adverse selection where agents exchange contracts specifying a price and a probability of trade, and shows that even with a complete contract structure, equilibria are not typically incentive-efficient.

As for the surplus realized in a decentralized market with stationary entry, recall that by Proposition 4 as frictions vanish the flow surplus approaches the surplus realized in the most efficient competitive equilibrium, $\sup S^C(q^H)$, and therefore compares with the surplus realized by an efficient mechanism as described above.

It's natural to wonder whether the superiority of decentralized trade when average quality is low is due to the fact that under decentralized trade units may trade at different times, while in the competitive model all units trade at one time. (In the literature studying the "mini-micro" foundations of competitive equilibrium, it is common to compare the outcomes generated in a *dynamic* decentralized market to those predicted by the *static* competitive model, thereby ignoring the role of time.) To address this issue, consider a market for lemons with one-time entry which operates over two periods. It is straightforward to extend the notion of competitive equilibrium to this dynamic setting, allowing units to trade at each of the two periods and allowing different prices at each period. (See Wooders (1998), for example, for a model of dynamic competitive equilibrium in a homogenous goods market.) It can be shown that when only low-quality units trade in the (unique) competitive equilibrium of the static market (that is, when $u(q^H) < c^H$), then for discount factors near one the dynamic competitive equilibrium has the same feature: only low-quality units trade. Thus, for discount factors near one the competitive surplus is the same for both the dynamic and the static market.[11]

For markets for lemons with stationary entry, Janssen and Roy (2000) have shown that the only stationary dynamic competitive equilibrium is the repetition of the static competitive equilibrium.[12] Thus, time alone does not explain the difference in surplus realized under centralized and decentralized trade.

[11] In fact, when $u(q^H) > c^H$ then a dynamic competitive market may yield even less surplus than the static competitive market.

[12] They also find non-stationary equilibria, however, where all qualities trade although with delay. The authors do not evaluate the surplus realized at these equilibria – they focus on the issue of price volatility.

5 Appendix: Proofs

Proof of Proposition 2: We begin by establishing a number of lemmas. Let $s^* = [(p_0^{B_i}, p_1^{B_i}; \lambda^i)_{i=1}^{n^B}, (r_0^H, r_1^H, 1), (r_0^L, r_1^L, 1)]$ be a market equilibrium. For $p \in \mathbb{R}_+$ define

$$P_t^B(p) = \sum_{\tau \in \{H,L\}} \mu_t^\tau [(u^\tau - p) I(p, r_t^\tau) + (1 - I(p, r_t^\tau)) \delta V_{t+1}^B];$$

i.e., $P_t^B(p)$ is the expected utility to a buyer matched at time t when play proceeds according to s^*, except that at time t the buyer proposes the price p. Note that by $ME.B$ we have for each $i \leq n^B$ that $P_t^B(p_t^{B_i}) \geq P_t^B(p)$. Hence $p_t^{B_{i'}} \neq p_t^{B_{i''}}$ implies $P_t^B(p_t^{B_{i'}}) = P_t^B(p_t^{B_{i''}})$.

Lemma 1. *For each $t \in \{0,1\}$ and $i \leq n^B$: $p_t^{B_i} \leq \max\{r_t^H, r_t^L\}$.*

Proof: Suppose that $p_t^{B_i} > \max\{r_t^H, r_t^L\}$ for some $i \leq n^B$ and $t \in \{0,1\}$. Then $I(p_t^{B_i}, r_t^H) = I(p_t^{B_i}, r_t^L) = 1$, and hence

$$P_t^B(p_t^{B_i}) = u(\mu_t^H) - p_t^{B_i}.$$

Let $p = \max\{r_t^H, r_t^L\}$; then $I(p, r_t^H) = I(p, r_t^L) = 1$, and therefore

$$P_t^B(p) = u(\mu_t^H) - p > u(\mu_t^H) - p_t^{B_i},$$

which contradicts $ME.B$. $\square$

Lemma 2. *For $t \in \{0,1\}$: $r_t^H = c^H > r_t^L$, and $Z_t^L \geq Z_t^H$. Moreover, $\mu_1^H \geq \mu_0^H$.*

Proof: By $ME.H$ and $ME.L$, we have $r_1^H = c^H > r_1^L = c^L$. Hence $Z_1^L \geq Z_1^H$. Also $p_1^{B_i} \leq c^H$ for each $i \leq n^B$ by Lemma 1, and therefore $V_1^H = 0$ and $V_1^L \leq \alpha(c^H - c^L)$. Thus, $r_0^H = c^H$ by $ME.H$, and $r_0^L \leq c^L + \delta\alpha(c^H - c^L) < c^H$ by $ME.L$. Hence $Z_0^L \geq Z_0^H$. Finally

$$\mu_1^H = \frac{(1 - \alpha Z_0^H)\mu_0^H}{(1 - \alpha Z_0^H)\mu_0^H + (1 - \alpha Z_0^L)\mu_0^L} \geq \frac{(1 - \alpha Z_0^H)\mu_0^H}{(1 - \alpha Z_0^H)\mu_0^H + (1 - \alpha Z_0^H)\mu_0^L} = \mu_0^H. \ \square$$

Lemma 3. *For each $i \leq n^B$ and $t \in \{0,1\}$, either $p_t^{B_i} = c^H$, or $p_t^{B_i} \leq r_t^L$.*

Proof: Since $p_t^{B_i} \leq c^H$ and $r_t^L < c^H$ by lemmas 1 and 2, assume by way of contradiction that $r_t^L < p_t^{B_i} < c^H$ for some $i \leq n^B$. Then $I(p_t^{B_i}, r_t^H) = 0$, $I(p_t^{B_i}, r_t^L) = 1$ and, since $\mu_t^L > 0$ (because $\alpha < 1$), we have

$$P_t^B(p_t^{B_i}) = \mu_t^L(u^L - p_t^{B_i}) + \mu_t^H \delta V_{t+1}^B < \mu_t^L(u^L - r_t^L) + \mu_t^H \delta V_{t+1}^{B_i} = P_t^B(r_t^L),$$

which contradicts $ME.B$. $\square$

Lemma 4. *For* $t \in \{0, 1\}$:

($L4.1$) $p_1^{B_i} \in \{c^H, c^L\}$ *for each* $i \leq n^B$, *and* $Z_1^L = 1$;

($L4.2$) *if* $\mu_t^H > q^*$, *then* $p_t^{B_i} = c^H$ *for each* $i \leq n^B$;

($L4.3$) *if* $\mu_t^H < q^*$ *and* $r_t^L = c^L$, *then* $p_t^{B_i} \leq c^L$ *for each* $i \leq n^B$.

Proof: Since $V_2^L = 0$, $ME.L$ implies $r_1^L = c^L$. In view of Lemma 3, in order to establish $L4.1$ we must show $p_1^{B_i} \geq c^L$. Suppose by way of contradiction that $p_1^{B_i} < c^L$; then (recall that $V_2^B = 0$)

$$P_1^B(p_1^{B_i}) = 0 < \mu_1^L(u^L - c^L) = P_1^B(r_1^L),$$

which contradicts $ME.B$. Since $p_1^{B_i} \in \{c^H, c^L\}$ for each $i \leq n^B$, we have $I(p_1^{B_i}, r_1^L) = 1$ for each $i \leq n^B$, and hence $Z_1^L = 1$.

We prove $L4.2$. Assume $\mu_t^H > q^*$ for some $t \in \{0, 1\}$. If $t = 1$, then $\mu_1^H > q^*$ implies

$$P_1^B(c^L) = \mu_1^L(u^L - c^L) < u(\mu_1^H) - c^H = P_1^B(c^H).$$

Hence $ME.B$ and $L4.1$ imply $p_1^{B_i} = c^H$ for each $i \leq n^B$. If $t = 0$, then $\mu_1^H \geq \mu_0^H > q^*$ by Lemma 2, and therefore $p_1^{B_i} = c^H$ for each $i \leq n^B$. Thus, $V_1^L = \alpha(c^H - c^L)$, $r_0^L = c^L + \alpha\delta(c^H - c^L)$, and $V_1^B = \alpha(u(\mu_1^H) - c^H)$. We have

$$P_0^B(c^H) = u(\mu_0^H) - c^H,$$

and

$$P_0^B(r_0^L) = \mu_0^L(u^L - r_0^L) + \mu_0^H \delta V_1^B.$$

To establish that $P_0^B(c^H) > P_0^B(r_0^L)$ it suffices to show, since $V_1^B \leq \alpha(u^H - c^H)$, that $u(\mu_0^H) - c^H > \mu_0^L(u^L - r_0^L) + \mu_0^H \delta\alpha(u^H - c^H)$. Note that $\mu_0^H > q^*$ implies

$$(1 - \delta\alpha)[\mu_0^H u^H + \mu_0^L c^L - c^H] > 0.$$

Adding $\mu_0^L u^L$ to each side of this inequality and rearranging the result yields

$$\mu_0^H u^H + \mu_0^L u^L - c^H > \mu_0^L(u^L - (c^L + \delta\alpha(c^H - c^L))) + \mu_0^H \delta\alpha(u^H - c^H),$$

which is the desired inequality. Hence $ME.B$ implies $p_0^{B_i} \neq r_0^L$ for each $i \leq n^B$, and Lemma 3 implies that either $p_0^{B_i} = c^H$ or $p_0^{B_i} < r_0^L$ for each $i \leq n^B$. Hence for each $i \leq n^B$, either $I(p_0^{B_i}, r_0^H) = I(p_0^{B_i}, r_0^L) = 1$ or $I(p_0^{B_i}, r_0^H) = I(p_0^{B_i}, r_0^L) = 0$, and so $Z_0^H = Z_0^L$. This implies

$$\mu_1^H = \frac{(1-\alpha Z_0^H)\mu_0^H}{(1-\alpha Z_0^H)\mu_0^H + (1-\alpha Z_0^L)\mu_0^L} = \mu_0^H.$$

Suppose that $p_0^{B_i} < r_0^L$ for some i. Then $\mu_1^H = \mu_0^H > q^*$ and $\delta\alpha < 1$ implies $\delta\alpha(u(\mu_1^H) - c^H) < u(\mu_0^H) - c^H$. Hence

$$P_0^B(p_0^{B_i}) = \delta V_1^B = \delta\alpha(u(\mu_1^H) - c^H) < u(\mu_0^H) - c^H = P_0^B(c^H),$$

which contradicts $ME.B$. Hence $p_0^{B_i} = c^H$ for each $i \leq n^B$.

We prove $L4.3$. Assume that $\mu_t^H < q^*$ and $r_t^L = c^L$. We show that $p_t^{B_i} < c^H = r_t^H$ for each $i \leq n^B$, which implies $p_t^{B_i} \leq r_t^L = c^L$ for each $i \leq n^B$ by Lemma 3, and establishes $L4.3$. Suppose by way of contradiction that for some $i \leq n^B$ we have $p_t^{B_i} = c^H$. Then $\mu_t^H < q^*$ and $\mu_t^H V_{t+1}^B \geq 0$ implies

$$P_t^B(c^H) = u(\mu_t^H) - c^H < \mu_t^L(u^L - c^L) + \mu_t^H \delta V_{t+1}^B = P_t^B(c^L),$$

which contradicts $ME.B$. Hence $p_t^{B_i} < c^H = r_t^H$ for each $i \leq n^B$. $\square$

Lemma 5. *Assume that $q^H < q^*$, and that α and δ are sufficiently close to 1 that (i) $\frac{q^H}{q^H+(1-\alpha)q^L} > q^*$, (ii) $\alpha\delta(c^H - c^L) > u^L - c^L$, and (iii) $\alpha\delta(u(q^*) - c^H) > u(q^H) - c^H$. Then*

(L5.1) *there are i', i'' such that $p_1^{B_{i'}} = c^H$ and $p_1^{B_{i''}} = c^L$.*

(L5.2) *$\mu_1^H = q^*$, and*

(L5.3) *$p_0^{B_i} \neq c^H$ for each $i \leq n^B$, $p_0^{B_k} = r_0^L$, and $p_0^{B_{k'}} < r_0^L$ for some k, k'.*

Proof: We show that $p_1^{B_i} = r_1^L = c^L$ for some i. Suppose not; then $L4.1$ implies $p_1^{B_i} = c^H$ for each $i \leq n^B$ and hence $V_1^B = \alpha(u(\mu_1^H) - c^H)$. Since $r_1^L = c^L$, we have $\mu_1^H \geq q^*$ by $L4.3$ and thus $V_1^B \geq \alpha(u(q^*) - c^H)$. By (iii) we have

$$P_0^B(c^H) = u(\mu_0^H) - c^H < \delta\alpha\left(u(q^*) - c^H\right) \leq \delta V_1^B,$$

and hence $p_0^{B_i} \neq c^H$ for each $i \leq n^B$. Since $p_1^{B_i} = c^H$ for each $i \leq n^B$ we have $r_0^L = c^L + \alpha\delta(c^H - c^L)$. By (ii) we have $u^L - r_0^L < 0$ and

$$P_0^B(r_0^L) = \mu_0^L(u^L - r_0^L) + \mu_0^H \delta V_1^B < \delta V_1^B.$$

Hence $p_0^{B_i} \neq r_0^L$ for each $i \leq n^B$. Lemma 3 therefore implies $p_0^{B_i} < r_0^L$ for each $i \leq n^B$. Thus $Z_0^H = Z_0^L = 0$, and therefore $\mu_1^H = \mu_0^H = q^H < q^*$, which is a contradiction.

Next we show that $p_1^{B_i} = r_1^H$ for some i. Suppose not; then $p_1^{B_i} = c^L$ for each $i \leq n^B$ by $L4.1$. Thus, $V_1^B = \alpha\mu_1^L(u^L - c^L)$, $V_1^L = 0$, and $r_0^L = c^L$. Since $\mu_0^H = q^H < q^*$, then $p_0^{B_i} \leq c^L$ for each $i \leq n^B$ by $L4.3$. For $p < c^L$ we have

$$P_0^B(p) = \delta V_1^B = \delta\alpha\mu_1^L(u^L - c^L) < \mu_0^L(u^L - c^L) + \mu_0^H \delta V_1^B = P_0^B(c^L),$$

where $P_0^B(p) < P_0^B(c^L)$ since $\mu_0^L \geq \mu_1^L$ (as $\mu_0^H \leq \mu_1^H$ by Lemma 2) and since $V_1^B \geq 0$. Therefore $p_0^{B_i} = c^L$ for each $i \leq n^B$ by $ME.B$. Hence $Z_0^H = 0$, and $Z_1^L = 1$. By (i) we have

$$\mu_1^H = \frac{\mu_0^H}{\mu_0^H + (1-\alpha)\mu_0^L} = \frac{q^H}{q^H + (1-\alpha)q^L} > q^*,$$

which implies by $L4.2$ that $p_1^{B_i} = c^H$ for each $i \leq n^B$, which is a contradiction. Therefore $p_1^{B_i} = r_1^H$ for some i.

Since $p_1^{B_i} = c^L$ for some i and $p_1^{B_i} = c^H$ for some i, then by $ME.B$ we have

$$P_1^B(c^L) = \mu_1^L(u^L - c^L) = u(\mu_1^H) - c^H = P_1^B(c^H),$$

which implies $\mu_1^H = q^*$. This establishes $L5.2$.

We establish $L5.3$. Since $p_1^{B_i} = c^H$ for some i by $L5.1$ and since $\mu_1^H = q^*$ by $L5.2$, we have $V_1^B = \alpha(u(q^*) - c^H)$. By Lemma 3 either $p_0^{B_i} = c^H$ or $p_0^{B_i} \leq r_0^L$. For $p < r_0^L$, (iii) implies

$$P_0^B(c^H) = u(\mu_0^H) - c^H = u(q^H) - c^H < \delta\alpha(u(q^*) - c^H) = \delta V_1^B = P_0^B(p).$$

Hence $ME.B$ implies $p_0^{B_i} \neq c^H$ for each $i \leq n^B$. Finally, we show $p_0^{B_i} = r_0^L$ for some i and $p_0^{B_i} < r_0^L$ for some i. Clearly $p_0^{B_i} = r_0^L$ for some i, for if $p_0^{B_i} < c^L$ for all i, then $Z_0^L = Z_0^H = 0$ and $\mu_1^H = \mu_0^H = q^H < q^*$, which contradicts $\mu_1^H = q^*$. Suppose $p_0^{B_i} = r_0^L$ for each i. Then

$$\mu_1^H = \frac{\mu_0^H}{\mu_0^H + (1-\alpha)\mu_0^L} = \frac{q^H}{q^H + (1-\alpha)q^L} > q^*,$$

by (i), which also contradicts $\mu_1^H = q^*$. $\square$

Proof of Proposition 2: By Lemma 2, $r_t^H = c^H > r_t^L$ for $t \in \{0,1\}$.

In order to prove $P2.1$, we show that every market equilibrium satisfies $n^B = 1$, $p_0^{B_1} = p_1^{B_1} = c^H$, and $r_0^L = c^L + \delta\alpha(c^H - c^L) > r_1^L = c^L$, and therefore that equilibrium is unique. Assume $q^H > q^*$. Then $\mu_0^H = q^H > q^*$, and therefore by $L4.2$ we have $p_0^{B_i} = c^H$ for each $i \leq n^B$, and by Lemma 2 and $L4.2$ again we have $p_1^{B_i} = c^H$ for each $i \leq n^B$. Hence $n^B = 1$. And since $p_t^{B_i} = c^H$, we have $Z_t^\tau = 1$ for $\tau \in \{H, L\}$ and $t \in \{0,1\}$. Thus $r_1^L = c^L$ by $ME.L$, and

$$r_0^L = c^L + \delta\alpha Z_1^H(c^H - c^L) = c^L + \delta\alpha(c^H - c^L).$$

Assume that $q^H < q^*$ and that α and δ are sufficiently close to 1 so that (i) $\frac{q^H}{q^H+(1-\alpha)q^L} > q^*$, (ii) $\alpha\delta(c^H - c^L) > u^L - c^L$, (iii) $\alpha\delta(u(q^*) - c^H) > u(q^H) - c^H$, and (iv) $\alpha > (q^* - q^H)/(q^* q^L)$. In order to prove $P2.2$ we show that in every market equilibrium $p_0^{B_i} \in [0, r_0^L]$ $\forall i \leq n^B$, with $\sum_{i \in \{i | p_0^{B_i} = r_0^L\}} \lambda^{B_i} = (q^* - q^H)/(\alpha q^* q^L) < 1$, and $p_1^{B_i} \in \{r_1^H, r_1^L\}$ with $\sum_{i \in \{i | p_1^{B_i} = r_1^H\}} \lambda_1^{B_i} = [1 - \delta\alpha(1 - q^*)]\frac{u^L - c^L}{\delta\alpha(c^H - c^L)} < 1$, and $r_0^L = c^L + [1 - \delta\alpha(1 - q^*)](c^H - c^L) > r_1^L = c^L$. Moreover, the probabilities of trade are uniquely determined by equations (1) to (3), and are given by

$$Z_0^H = 0 < Z_0^L = \frac{q^* - \mu_0^H}{\alpha q^* \mu_0^L} = \frac{q^* - q^H}{\alpha q^* q^L},$$

and

$$Z_1^H = [1 - \delta\alpha(1 - q^*)]\frac{u^L - c^L}{\delta\alpha(c^H - c^L)} < Z_1^L = 1.$$

By Lemma 3 and $L5.3$ we have $p_0^{B_i} \leq r_0^L$ for each $i \leq n^B$. Hence $Z_0^H = 0$. Thus by $L5.2$ we have

$$\mu_1^H = q^* = \frac{\mu_0^H}{\mu_0^H + (1 - \alpha Z_0^L)\mu_0^L} = \frac{q^H}{q^H + (1 - \alpha Z_0^L)q^L},$$

and therefore

$$Z_0^L = \frac{q^* - \mu_0^H}{\alpha q^* \mu_0^L} = \frac{q^* - q^H}{\alpha q^* q^L}.$$

Because $q^H < q^*$ and (iv) above we have $0 < Z_0^L < 1$. Hence $p_0^{B_i} = r_0^L$ for some i and $p_0^{B_i} < r_0^L$ for some i, and therefore $ME.B$ implies

$$P_0^B(r_0^L) = \mu_0^L(u^L - r_0^L) + (1 - \mu_0^L)\delta V_1^B = \delta V_1^B.$$

Thus, $u^L - r_0^L = \delta V_1^B$, and since $p_1^{B_i} \in \{c^H, c^L\}$ for each $i \leq n^B$ by $L4.1$, we have $r_0^L = c^L + \delta V_1^L = c^L + \delta\alpha Z_1^H(c^H - c^L)$. Also $V_1^B = \alpha\mu_1^L(u^L - c^L) = \alpha(1-q^*)(u^L - c^L)$ since $\mu_1^H = q^*$ by $L5.2$. Thus

$$u^L - c^L - \delta\alpha Z_1^H(c^H - c^L) = \delta\alpha(1-q^*)(u^L - c^L).$$

Rearranging yields

$$Z_1^H = [1 - \delta\alpha(1-q^*)]\frac{u^L - c^L}{\delta\alpha(c^H - c^L)},$$

as the proportion of buyers offering $p_1^{B_i} = r_1^H$. Finally, $p_1^{B_i} \in \{c^H, c^L\}$ is $L4.1$. Now, $r_1^L = c^L$ follows from $ME.L$, and therefore r_0^L is readily calculated as

$$r_0^L = c^L + \delta\alpha Z_1^H(c^H - c^L) = [1 - \delta\alpha(1-q^*)](u^L - c^L). \ \square$$

Proof of Proposition 3: Assume $q^H > q^*$. Then by $P2.1$ the market equilibrium satisfies $n^B = 1$ and

$$p_0^{B_1} = p_1^{B_1} = c^H = r_0^H = r_1^H > r_0^L = c^L + \delta\alpha(c^H - c^L) > r_1^L = c^L.$$

Hence equations (1) to (5) yield $\mu_t^\tau = q^\tau$ for $\tau \in \{H, L\}$ and $t \in \{0,1\}$, and Equation (6) yields

$$V_1^B = \alpha(\mu_0^H u^H + \mu_0^L u^L - c^H) = \alpha(q^H u^H + q^L u^L - c^H),$$

and

$$\begin{aligned} V_0^B &= \alpha(\mu_0^H u^H + \mu_0^L u^L - c^H) + (1-\alpha)\delta V_1^B \\ &= \alpha(1 + (1-\alpha)\delta)(q^H u^H + q^L u^L - c^H). \end{aligned}$$

For sellers, Equation (7) yields $V_1^H = V_0^H = 0$, $V_1^L = \alpha(c^H - c^L)$, and

$$V_0^L = \alpha(c^H - c^L) + (1-\alpha)\delta V_1^L = \alpha(1 + (1-\alpha)\delta)(c^H - c^L).$$

Thus, the surplus is

$$S^D(q^H) = \alpha(1 + (1-\alpha)\delta)[q^H(u^H - c^H) + q^L(u^L - c^L)],$$

and therefore

$$\tilde{S}^D(q^H) = q^H(u^H - c^H) + q^L(u^L - c^L).$$

Hence for δ and α near one we have

$$q^L(u^L - c^L) = \inf S^C(q^H) < S^D(q^H) < \tilde{S}^D(q^H) = \sup S^C(q^H),$$

and therefore $P3.1$ holds.

Assume $q^H < q^*$. We compute V_0^B. By $P2.2$, $p_1^B = c^H = r_1^H > r_1^L$ is an optimal price offer, and $\mu_1^H = q^*$; hence Equation (6) yields

$$V_1^B = \alpha(q^* u^H + (1 - q^*)u^L - c^H).$$

And since offering $p_0^B < r_0^L$ is optimal by $P2.2$, Equation (6) again yields

$$V_0^B = \delta V_1^B = \delta\alpha(q^* u^H + (1 - q^*)u^L - c^H).$$

For sellers, $P2.2$ and Equation (7) yield $V_1^H = V_0^H = 0$. For low-quality sellers, as shown in the proof of Proposition 2, we have $p_1^{B_i} \in \{c^H, c^L\}$ and $\sum_{i\in\{i|p_1^{B_i}=r_1^H\}} \lambda_1^{B_i} = [1 - \delta\alpha(1 - q^*)]\frac{u^L - c^L}{\delta\alpha(c^H - c^L)}$. Hence Equation (7) yields

$$V_1^L = \alpha(1 - \delta\alpha(1 - q^*))\frac{u^L - c^L}{\delta\alpha(c^H - c^L)}(c^H - c^L).$$

Also $P2.2$ we have $p_0^{B_i} \leq r_0^L$ for $i \leq n^B$. Hence

$$V_0^L = \delta V_1^L = (1 - \delta\alpha(1 - q^*))(u^L - c^L).$$

Hence the surplus is

$$\begin{aligned} S^D(q^H) &= \delta\alpha(q^* u^H + (1 - q^*)u^L - c^H) + q^L(1 - \delta\alpha(1 - q^*))(u^L - c^L) \\ &= q^L(u^L - c^L) + q^H(u^H - c^H)\alpha\delta\frac{u^L - c^L}{u^H - c^L}, \end{aligned}$$

and therefore

$$\tilde{S}^D(q^H) = q^L(u^L - c^L) + q^H(u^H - c^H)\frac{u^L - c^L}{u^H - c^L}.$$

Thus, for δ and α near one, we have $S^D(q^H) < \tilde{S}^D(q^H)$.

If $u(c^H) > q^H$ then $\inf S^C(q^H) = q^L(u^L - c^L)$ and $\sup S^C(q^H) = q^L(u^L - c^L) + q^H(u^H - c^H)$. Thus

$$\inf S^C(q^H) < S^D(q^H) < \tilde{S}^D(q^H) < \sup S^C(q^H),$$

and therefore $P3.2$ holds.

If $u(q^H) < c^H$ then $\inf S^C(q^H) = \sup S^C(q^H) = q^L(u^L - c^L)$ and $q^H < q^*$, and therefore

$$\inf S^C(q^H) = \sup S^C(q^H) < S^D(q^H) < \tilde{S}^D(q^H).$$

Thus, $P3.3$ holds. □

Proof of Proposition 4: Assume that $\delta < 1$ is sufficiently near one that

$$\alpha\delta(c^H - c^L) > (1 - \delta(1 - \alpha))(u^L - c^L).$$

(Since $c^H > u^L$ the above inequality holds for $\delta = 1$, so it also holds for δ near one.)

We prove $P4.1$. Assume that $u(q^H) \geq c^H$. We show that $n^B = 1$, $p^B = c^H$, $r^H = c^H$, and $r^L = c^L + \delta\alpha(c^H - c^L)/(1 - \delta(1 - \alpha))$ is a stationary equilibrium. For $\tau \in \{H, L\}$ let $K^\tau = q^\tau/\alpha$. Then $\mu^\tau = q^\tau$. Since all buyers offer $p^B = c^H$, the sellers' expected utilities are, respectively, $V^H = 0$, and

$$V^L = \alpha(c^H - c^L) + \delta(1 - \alpha)V^L = \frac{\alpha(c^H - c^L)}{1 - \delta(1 - \alpha)}.$$

Hence $SE.H$ and $SE.L$ hold. Also $I(p^B, r^H) = I(p^B, r^L) = 1$ and therefore conditions $SE.K$ is satisfied. For the given strategy distribution, since $p^B = c^H = r^H > r^L$ the buyers' expected utility is

$$V^B = \alpha(u(\mu^H) - c^H) + \delta(1 - \alpha)V^B = \frac{\alpha(u(\mu^H) - c^H)}{1 - \delta(1 - \alpha)} \geq 0.$$

Since $\mu^\tau = q^\tau$ we need to show that

$$r^H \in \arg\max_p \sum_{\tau \in \{H,L\}} q^\tau[I(p, r^\tau)(u^\tau - p) + (1 - I(p, r^\tau))\delta\frac{\alpha(u(q^H) - c^H)}{1 - \delta(1 - \alpha)}].$$

Clearly, the arg max satisfies either $p = c^H$, $p = r^L$, or $p < r^L$. With $p = c^H$ the value of the objective function is $u(q^H) - c^H \geq 0$; with $p = r^L$ the value of the objective function is

$$q^H\delta\frac{\alpha(u(q^H) - c^H)}{1 - \delta(1 - \alpha)} + q^L(u^L - r^L).$$

This is clearly less than $u(q^H)-c^H$ since $u^L-r^L<0$ follows from $\delta\alpha(c^H-c^L)>(1-\delta(1-\alpha))(u^L-c^L)$. Finally, for $p<r^L$ the value of the objective is $\delta\alpha(u(q^H)-c^H)/(1-\delta(1-\alpha))$ which is less than or equal to $u(q^H)-c^H$. Hence offering $p^B=r^H=c^H$ is optimal, and therefore $SE.B$ is satisfied.

In order to complete the proof of $P4.1$ we compute the flow surplus. Noticing that $\mu^\tau=q^\tau$ for $\tau\in\{H,L\}$, we have

$$\begin{aligned}S^F(q^H) &= V^B+q^HV^H+q^LV^L\\ &= \frac{\alpha(u(q^H)-c^H)}{1-\delta(1-\alpha)}+q^L\frac{\alpha(c^H-c^L)}{1-\delta(1-\alpha)}\\ &= \frac{\alpha}{1-\delta(1-\alpha)}(q^H(u^H-c^H)+q^L(u^L-c^L)).\end{aligned}$$

Thus $S^F(q^H)<\sup S^C(q^H)$, and $\lim_{\delta\to1}S^F(q^H)=\sup S^C(q^H)$. We also have $\lim_{\delta\to1}V^H=0$, $\lim_{\delta\to1}V^L=c^H-c^L$, and $\lim_{\delta\to1}V^H=u(q^H)-c^H$. Hence each type of trader obtains the same payoff as in the competitive equilibrium in which the price is c^H and all units trade (see Proposition 1.1).

We prove $P4.2$. Assume now that $u(q^H)<c^H$. We show that the strategy distribution given by $n^H=3$, $p^{B_1}=c^H$, $p^{B_2}=u^L$, $p^{B_3}<u^L$, $\lambda^{B_1}=(1-\delta)(u^L-c^L)/[\alpha\delta(c^H-u^L)]$, $\lambda^{B_2}=\lambda^{B_1}(c^H-u(q^H))/[q^H(c^H-u^L)]$, $\lambda^{B_3}=1-\lambda^{B_1}-\lambda^{B_2}$, $r^H=c^H$, and $r^L=u^L$ is a stationary equilibrium. Note that since $c^H>u^L>c^L$ then $\lambda^{B_1}>0$; and since $c^H-u(q^H)>0$ then $\lambda^{B_2}>0$. Moreover, for δ sufficiently close to one we have

$$\lambda^{B_1}+\lambda^{B_2}=\frac{(1-\delta)(u^L-c^L)}{\alpha\delta(c^H-u^L)}\left(1+\frac{c^H-u(q^H)}{q^H(c^H-u^L)}\right)<1,$$

and therefore $\lambda^{B_3}>0$.

For the stocks given by

$$K^H=\frac{q^H}{\lambda^{B_1}\alpha},$$

and

$$K^L=\frac{q^L}{(\lambda^{B_1}+\lambda^{B_2})\alpha},$$

conditions $SE.K$ is satisfied. Also, since $p^{B_i}\le c^H$ for $i\le n^B$, we have $V^H=0$, and therefore $SE.H$ holds. As for low-quality sellers we have

$$V^L=\alpha\lambda^{B_1}(c^H-c^L)+(1-\alpha\lambda^{B_1})\delta V^L=\frac{\alpha\lambda^{B_1}(c^H-c^L)}{1-(1-\alpha\lambda^{B_1})\delta}=\frac{u^L-c^L}{\delta},$$

and therefore $SE.L$ holds. For buyers, since

$$\mu^H = \frac{K^H}{K^H + K^L} = \frac{q^H}{q^H + \frac{q^L \lambda^{B_1}}{\lambda^{B_1} + \lambda^{B_2}}} = \frac{c^H - u^L}{u^H - u^L},$$

we have $u(\mu^H) = c^H = r^H$. Thus, since $u^L = r^L$, we have $V^B = 0$, and therefore $SE.B$ holds.

In order to complete the proof of $P5.2$ we compute the (flow) surplus.

$$S^F(q^H) = V^B + q^H V^H + q^L V^L = \frac{q^L(u^L - c^L)}{\delta}.$$

Thus, we have both $S^F(q^H) > S^C(q^H)$ and $\lim_{\delta\to 1} S^F(q^H) = S^C(q^H)$. We also have $\lim_{\delta\to 1} V^H = 0$, $\lim_{\delta\to 1} V^L = u^L - c^L$, and $\lim_{\delta\to 1} V^H = 0$. Hence each type of trader obtains the same payoff as the (unique) competitive equilibrium.

Proof of P4.3: In order to establish ($P4.3$) we first prove a number of intermediate facts. Let $[(p^{B_i}, \lambda^{B_i})_{i=1}^{n^B}, r^H, r^L]$ be a stationary equilibrium.

Lemma 6. *For $p \in \mathbb{R}_+$ we have*

$$I(p, r^H)(p - c^H) + (1 - I(p, r^H))\delta V^H \geq I(p, r^L)(p - c^H) + (1 - I(p, r^L))\delta V^H.$$

Proof: We show for any p that

$$(I(p, r^H) - I(p, r^L))(p - c^H) \geq (I(p, r^H) - I(p, r^L))\delta V^H.$$

The inequality trivially holds for p such that $I(p, r^H) - I(p, r^L) = 0$. If $I(p, r^H) - I(p, r^L) = 1$ then $p \geq r^H$. Hence $p - c^H \geq r^H - c^H = \delta V^H$ and the inequality holds. If $I(p, r^H) - I(p, r^L) = -1$ then $p < r^H$ and hence $p - c^H < r^H - c^H = \delta V^H$ and so $-(p - c^H) > -\delta V^H$. □

For $\tau \in \{H, L\}$, write $Z^\tau = \sum_{i=1}^{n^B} \lambda^{B_i} I(p^{B_i}, r^\tau)$.

Lemma 7: ($L7.1$) $r^H > r^L$, *and* ($L7.2$) $V^L - V^H < c^H - c^L$.

Proof: ($L7.2$) implies ($L7.1$) since $c^H - c^L > \delta(V^L - V^H) \iff c^H + \delta V^H > c^L + \delta V^L \iff r^H > r^L$. We establish ($L7.2$). We have that

$$V^H = \alpha \sum_{i=1}^{n^B} \lambda^{B_i}[I(p^{B_i}, r^H)(p^{B_i} - c^H) + (1 - I(p^{B_i}, r^H))\delta V^H] + \delta(1 - \alpha)V^H,$$

and hence by Lemma 6 we have

$$V^H \geq \alpha \sum_{i=1}^{n^B} \lambda^{B_i}[I(p^{B_i}, r^L)(p^{B_i} - c^L + c^L - c^H) + (1 - I(p^{B_i}, r^L))\delta V^H] + \delta(1-\alpha)V^H.$$

Rewriting yields

$$V^H \geq \alpha Z^L(c^L - c^H) + \alpha \sum_{i=1}^{n^B} \lambda^{B_i} I(p^{B_i}, r^L)(p^{B_i} - c^L) + (1 - \alpha Z^L)\delta V^H;$$

i.e.,

$$V^H \geq \frac{\alpha}{1 - \delta(1 - \alpha Z^L)}[Z^L(c^L - c^H) + \sum_{i=1}^{n^B} \lambda^{B_i} I(p^{B_i}, r^L)(p^{B_i} - c^L)].$$

Since

$$\begin{aligned} V^L &= \alpha \sum_{i=1}^{n^B} \lambda^{B_i} I(p^{B_i}, r^L)(p^{B_i} - c^L) + (1 - \alpha Z^L)\delta V^L \\ &= \frac{\alpha}{1 - \delta(1 - \alpha Z^L)} \sum_{i=1}^{n^B} \lambda^{B_i} I(p^{B_i}, r^L)(p^{B_i} - c^L), \end{aligned}$$

we have

$$V^H \geq \frac{\alpha Z^L(c^L - c^H)}{1 - \delta(1 - \alpha Z^L)} + V^L.$$

Since $\frac{\alpha Z^L}{1-\delta(1-\alpha Z^L)} < 1$ for $\delta < 1$ and since $c^L - c^H < 0$, we have $V^H - V^L > c^L - c^H$ or $V^L - V^H < c^H - c^L$. $\square$

Lemma 8: *For each $i \leq n^B$, either $p^{B_i} = r^H$ or $p^{B_i} \leq r^L$.*

Proof: Clearly $p^{B_i} \leq r^H$. For $p = r^L$ we have $I(p, r^L) = 1$ and $I(p, r^H) = 0$, and therefore

$$\sum_{\tau \in \{H,L\}} \mu^\tau[(u^\tau - p)I(p, r^\tau) + (1 - I(p, r^\tau))\delta V^B] = \mu^L(u^L - p) + (1 - \mu^L)\delta V^B.$$

For $r^L < p' < r^H$ we have $I(p', r^L) = 1$ and $I(p', r^H) = 0$, and therefore

$$\begin{aligned} \sum_{\tau \in \{H,L\}} \mu^\tau[(u^\tau - p')I(p', r^\tau) + (1 - I(p', r^\tau))\delta V^B] &= \mu^L(u^L - p') + (1 - \mu^L)\delta V^B \\ &< \mu^L(u^L - p) + (1 - \mu^L)\delta V^B. \end{aligned}$$

Hence p' does not satisfy $SE.B$. Thus $p^{B_i} \notin (r^L, r^H)$ for $i \leq n^B$. This establishes the claim. $\square$

Lemma 9: $(L9.1)$ $r^H = c^H$, *and* $(L9.2)$ $V^H = 0$.

Proof: We prove $(L9.2)$, which implies $(L9.1)$ since $r^H - c^H = \delta V^H$ by $SE.H$. By Lemma 8 we have $p^{B_i} \leq r^H$ for $i \leq n^B$; hence

$$\begin{aligned} V^H &= \frac{\alpha}{1-\delta(1-\alpha Z^H)} \sum_{i=1}^{n^B} \lambda^{B_i} I(p^{B_i}, r^H)(p^{B_i} - c^H) \\ &= \frac{\alpha Z^H (r^H - c^H)}{1-\delta(1-\alpha Z^H)} = \frac{\alpha \delta Z^H V^H}{1-\delta(1-\alpha Z^H)}; \end{aligned}$$

i.e.,

$$(1 - \frac{\alpha \delta Z^H}{1-\delta(1-\alpha Z^H)})V^H = 0.$$

Since $\frac{\alpha Z^H}{1-\delta(1-\alpha Z^H)} < 1$, we have $V^H = 0$. $\square$

Lemma 10: $(L10.1)$ $K^\tau > 0$ *and* $Z^\tau > 0$ *for* $\tau \in \{H, L\}$, $(L10.2)$ $Z^H \leq Z^L$, *and* $(L10.3)$ $\mu^H \geq q^H$.

Proof: By $SE.K$ we have $\alpha K^\tau Z^\tau = \alpha K^\tau \sum_{i=1}^{n^B} \lambda^{B_i} I(p^{B_i}, r^\tau) = q^\tau > 0$, and hence $K^\tau > 0$ and $Z^\tau > 0$ for $\tau \in \{H, L\}$. Thus $(L10.1)$ holds. As for $(L10.2)$, it is a direct implication of $L7.1$. Now by $(L10.2)$ we have

$$\mu^H = \frac{K^H}{K^H + K^L} = \frac{\frac{q^H}{\alpha Z^H}}{\frac{q^H}{\alpha Z^H} + \frac{q^L}{\alpha Z^L}} = \frac{q^H Z^L}{q^H Z^L + q^L Z^H} \geq \frac{q^H Z^L}{q^H Z^L + q^L Z^L} = q^H.$$

Hence $(L10.3)$ holds. $\square$

Lemma 11: *Assume that* $u^H - c^H > u^L - c^L$. *Then there is a* $\hat{\delta} < 1$ *such that for every* $1 > \delta > \hat{\delta}$ *either (i)* $n^B = 1$ *and* $p^{B_1} = r^H$, *or (ii) there is* $i \leq n^B$ *such that* $p^{B_i} < r^L$.

Proof: Suppose by way of contradiction that for every $\hat{\delta}$ there is $1 > \delta > \hat{\delta}$ such that $p^{B_i} \in \{r^H, r^L\}$ for $i \leq n^B$, $p^{B_i} = r^H$ for some i, and $p^{B_i} = r^L$ for some i. Assume, without loss of generality, that $p^{B_1} = r^H$ and $p^{B_2} = r^L$. Then

$$V^L = \alpha \lambda^{B_1}(r^H - c^L) + \alpha \lambda^{B_2}(r^L - c^L) + (1 - \alpha(\lambda^{B_1} + \lambda^{B_2}))\delta V^L.$$

Since $r^H = c^H$ by $(L9.1)$ and since $r^L - c^L = \delta V^L$ by $SE.L$, we obtain

$$V^L = \frac{\alpha\lambda^{B_1}(c^H - c^L)}{1 - \delta(1 - \alpha\lambda^{B_1})}.$$

Therefore since $u^L \geq r^L$ (because $p^{B_i} = r^L$ for some i) and $r^L = c^L + \delta V^L$, we have

$$c^L + \delta\frac{\alpha\lambda^{B_1}(c^H - c^L)}{1 - \delta(1 - \alpha\lambda^{B_1})} \leq u^L.$$

Rearranging yields

$$\lambda^{B_1} \leq \frac{(1-\delta)(u^L - c^L)}{\delta\alpha(c^H - u^L).}$$

Note that the bound on λ^{B_1} can be made arbitrarily small by choosing δ sufficiently close to 1. Furthermore, since

$$\mu^H = \frac{q^H Z^L}{q^H Z^L + q^L Z^H} = \frac{q^H}{q^H + q^L\lambda^{B_1}},$$

then μ^H can be made arbitrarily close to 1 for δ sufficiently close to 1. Hence, using $u^H - c^H > u^L - c^L$, there is a $\hat{\delta} < 1$ such that $\delta > \hat{\delta}$ implies

$$\mu^H u^H + \mu^L u^L - c^H > u^L - c^L.$$

Since $r^L \geq c^L$ this implies for $\delta > \hat{\delta}$ that

$$\mu^H u^H + \mu^L u^L - c^H > u^L - r^L.$$

Furthermore, since $p^{B_1} = r^H$ is an optimal offer we have

$$\mu^H u^H + \mu^L u^L - c^H \geq \delta V^B.$$

Hence $\mu^L > 0$ implies

$$\mu^H u^H + \mu^L u^L - c^H > \mu^H \delta V^B + \mu^L(u^L - r^L),$$

but this contradicts that $p^{B_2} = r^L$ is an optimal price offer. $\square$

Proof of P4.3: (A) Assume that $u(q^H) > c^H$ and $u^H - c^H > u^L - c^L$. We first show that $p^{B_i} \geq r^L$ for each $i \leq n^B$. We have

$$\begin{aligned}
V^B &= \alpha \sum_{\tau\in\{H,L\}} \mu^\tau[I(p^{B_i}, r^\tau)(u^\tau - p^{B_i}) + (1 - I(p^{B_i}, r^\tau))\delta V^B] + \delta(1-\alpha)V^B \\
&= \frac{\alpha}{(1 - \delta(1 - \alpha\sum_{\tau\in\{H,L\}}\mu^\tau I(p^{B_i}, r^\tau))} \sum_{\tau\in\{H,L\}} \mu^\tau I(p^{B_i}, r^\tau)(u^\tau - p^{B_i}).
\end{aligned}$$

Assume that $p^{B_i} < r^L$ for some $i \leq n^B$. Then $I(p^{B_i}, r^H) = I(p^{B_i}, r^L) = 0$, and therefore

$$\sum_{\tau \in \{H,L\}} \mu^\tau [I(p^{B_i}, r^\tau)(u^\tau - p^{B_i}) + (1 - I(p^{B_i}, r^\tau))\delta V^B] = 0.$$

But $u(q^H) > c^H$ and $\mu^H \geq q^H$, by $L10.3$, implies $u(\mu^H) > c^H$. Hence for $p = r^H = c^H$ we have $I(p, r^H) = I(p, r^L) = 1$, and

$$\sum_{\tau \in \{H,L\}} \mu^\tau [I(p, r^\tau)(u^\tau - p) + (1 - I(p, r^\tau))\delta V^B] = u(\mu^H) - c^H > 0.$$

Then $p^{B_i} < r^L$ does not satisfy $SE.B$. Hence by Lemma 8 we have $p^{B_i} \in \{r^H, r^L\}$, and therefore by Lemma 11 there is a $\hat{\delta} < 1$ such that for every $\delta > \hat{\delta}$ we have $n^B = 1$ and $p^{B_1} = r^H$. Hence $r^H = c^H$ and $r^L = c^L + \delta\alpha(c^H - c^L)/(1 - \delta(1 - \alpha))$. $\square$

(B) Assume that $u(q^H) < c^H$ and $u^H - c^H > u^L - c^L$. First we show that $\mu^H > q^H$. Since $Z^H > 0$ (by $L10.1$) then $p^{B_i} = r^H$ for some $i \leq n^B$. For $\hat{p} < r^L$, $SE.B$ implies

$$\begin{aligned} & \sum_{\tau \in \{H,L\}} \mu^\tau [(u^\tau - r^H) I(r^H, r^\tau) + (1 - I(r^H, r^\tau))\delta V^B] \\ \geq & \sum_{\tau \in \{H,L\}} \mu^\tau [(u^\tau - \hat{p}) I(\hat{p}, r^\tau) + (1 - I(\hat{p}, r^\tau))\delta V^B], \end{aligned}$$

and since $r^H = c^H > r^L > \hat{p}$, we have

$$\sum_{\tau \in \{H,L\}} \mu^\tau (u^\tau - c^H) \geq \delta V^B.$$

Thus, since $V^B \geq 0$ and $\sum_{\tau \in \{H,L\}} q^\tau (u^\tau - c^H) < 0$, this implies $\mu^H > q^H$.

We now establish that $p^{B_i} = r^L$ for some i. We have

$$\mu^H = \frac{q^H Z^L}{q^H Z^L + q^L Z^H} > q^H,$$

which implies $Z^L > q^H Z^L + q^L Z^H$. Since $Z^L \geq Z^H$ by ($L10.2$), we must have $Z^L > Z^H$. Hence there is an i such that $p^{B_i} = r^L$. By Lemma 11 there is a $\hat{\delta} < 1$ such that for every $\delta > \hat{\delta}$ we have $p^{B_i} < r^L$ for some i.

We have established that $p^{B_i} = r^H$ for some i, $p^{B_i} = r^L$ for some i, and $p^{B_i} < r^L$ for some i. Assume without loss of generality that $p^{B_1} = r^H$, $p^{B_2} = r^L$, and $p^{B_3} < r^L$.

Since $p^{B_3} < r^L$ is an optimal price offer, we have

$$V^B = \sum_{\tau \in \{H,L\}} \mu^\tau [(u^\tau - r^\tau) I(p^{B_3}, r^\tau) + (1 - I(p^{B_3}, r^\tau)) \delta V^B] + \delta(1-\alpha) V^B = \delta V^B$$

and because $\delta < 1$ this implies $V^B = 0$. Since the expected payoff to a buyer is equal for all three price offers we have

$$\mu^H (u^H - c^H) + \mu^L (u^L - c^H) = u^L - r^L = 0,$$

which implies $\mu^H = (c^H - u^L)/(u^H - u^L)$ and $r^L = u^L$. Hence

$$r^L = c^L + \delta \frac{\alpha \lambda^{B_1} (r^H - c^L)}{1 - \delta(1 - \alpha \lambda^{B_1})} = u^L,$$

which implies

$$\lambda^{B_1} = \frac{(1-\delta)(u^L - c^L)}{\delta \alpha \left(c^H - u^L\right)}.$$

Furthermore, since

$$\mu^H = \frac{K^H}{K^H + K^L} = \frac{q^H Z^L}{q^H Z^L + q^L Z^H} = \frac{q^H (\lambda^{B_1} + \lambda^{B_2})}{q^H (\lambda^{B_1} + \lambda^{B_2}) + q^L \lambda^{B_1}},$$

we have

$$\lambda^{B_2} = \lambda^{B_1} \frac{c^H - u(q^H)}{q^H (u^H - c^H)}.$$

The proof of $P4.1$ established that $\lambda^{B_1} > 0$, $\lambda^{B_2} > 0$, and $\lambda^{B_1} + \lambda^{B_2} < 1$ for δ near one. $\square$

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Figure 1
Competitive Equilibria

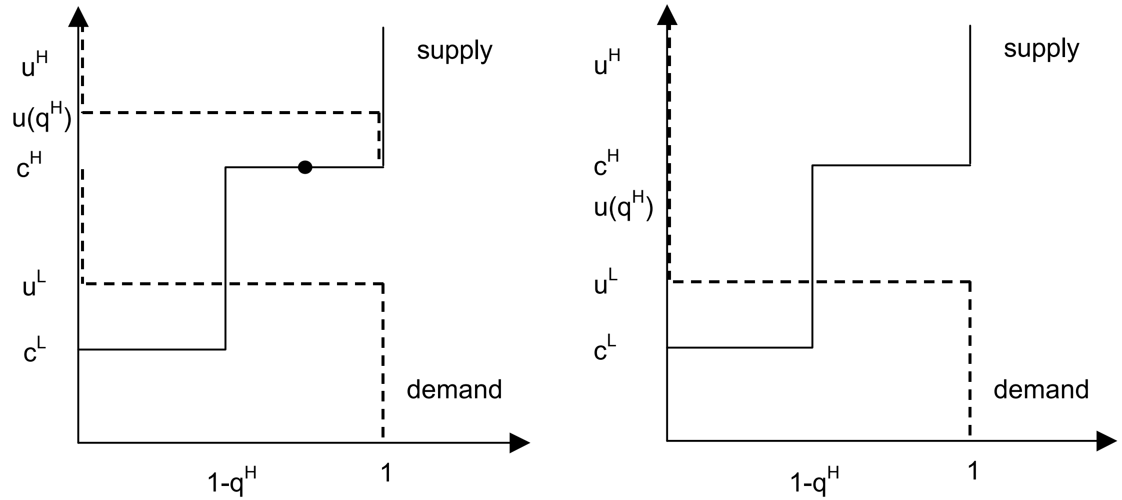

Figure 1(a): $u(q^H)>c^H>u^L$

Figure 1(b): $c^H>u(q^H)>u^L$

Figure 2

Surplus (as frictions vanish) under centralized and decentralized trade

$u^H=1$, $c^H=.5$, $u^L=.2$, $c^L=0$

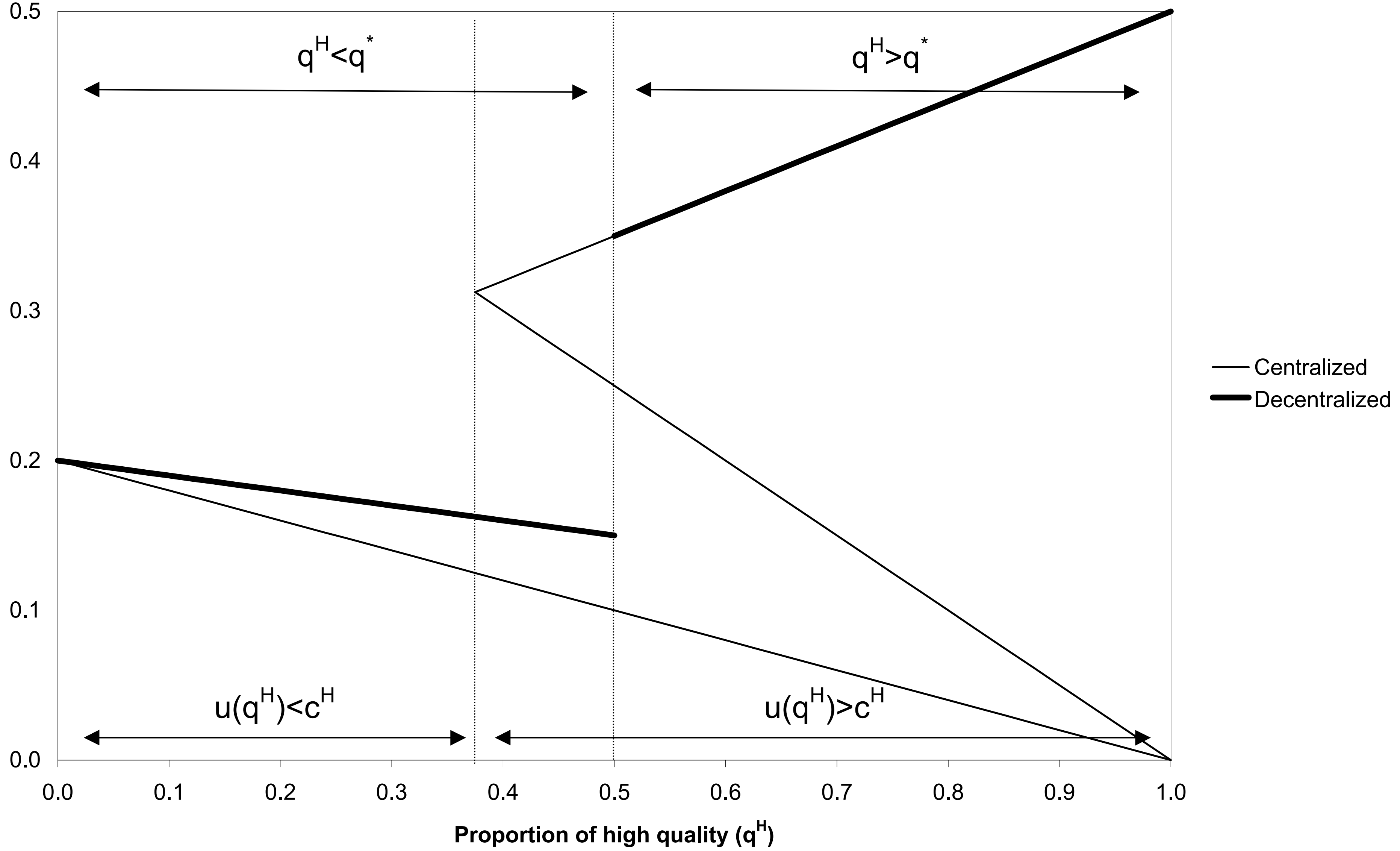

Figure 3

Efficient surplus and surplus under centralized and decentralized trade

$u^H=1$, $c^H=.5$, $u^L=.2$, $c^L=0$